

Learning seminar on BCHM

Lecture 8 (22 October 2021)

Finiteness of models
& finite generation

Emelie Arvidsson, institute for
advanced study.

Today we will prove things
 $A \sim E$.

Hacon McKernan

$$\begin{array}{l} \bullet F_{n-1} \xrightarrow{\downarrow} A_n \\ \bullet E_{n-1} \xrightarrow{\quad} B_n \end{array} \left. \begin{array}{l} \\ \end{array} \right\} A_b \rightarrow B_n \Rightarrow C_n$$

$$D_{n-1} + B_n + C_n \rightsquigarrow D_n$$

Today we will show

$$\begin{array}{l} C_n + D_n \rightsquigarrow E_n \\ C_n + D_n + E_n \Rightarrow F_n \end{array} \left. \begin{array}{l} \\ \end{array} \right\}$$

Theorem 1.2

Let (X, Δ) be a klt pair, where $K_X + \Delta$ is \mathbb{R} -Cartier. Let $\pi: X \rightarrow U$ be a projective morphism of quasi-projective varieties. If either

- Δ is π -big & $K_X + \Delta$ is pseudo-effective, or
 - $K_X + \Delta$ is π -big, then

(1) $K_X + \Delta$ has a log-terminal model over U

(2) If $K_X + \Delta$ is π -big then $K_X + \Delta$ has a log-canonical model over U

(3) If $K_X + \Delta$ is \mathbb{Q} -Cartier then the \mathcal{O}_U -algebra

$$R(X, \pi) := \bigoplus_{m \in \mathbb{N}} \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor)$$

is finitely generated.

Proof (Follows from theorems C-D)

If $K_X + \Delta$ is \mathbb{R} -big then $\exists B \geq 0$

$K_X + \Delta \sim_{\mathbb{R}, \text{lt}} B$ and so for $\epsilon > 0$

small $K_X + \Delta + \epsilon B$ is lt

$$K_X + \underbrace{\Delta + \epsilon B}_{\text{big}} \sim_{\mathbb{R}, \text{lt}} (1 + \epsilon) K_X + \Delta$$

16-models \hookrightarrow lt-models

w.l.o.g $K_X + \Delta$ pseudo-effective
 Δ big.

Theorem D (non-vanishing)

$$\Rightarrow \exists B \geq 0 \quad K_X + \Delta \sim_{\mathbb{R}, \text{lt}} B$$

Theorem C (\exists of 16-models)

in the situation above the
 $K_X + \Delta$ has a log terminal
model.

\Rightarrow (1) is ok.

From CDP

$\phi: X \dashrightarrow Y$ be a b>
 formal model for $K_X + \Delta$
 $\phi_* \Delta \simeq \Gamma$ then $K_Y + \Gamma$ is nef,
 Γ is big \Rightarrow $K_Y + \Gamma$ is
basepoint
free thus semisimple.

let $X \xrightarrow{\phi} Y \xrightarrow{t} Z$

b. ample model then

if $K_X + \Delta$ is big \Rightarrow
 $K_Y + \Gamma$ is big \Rightarrow $t \circ \phi$ is
 a lc-model. (2)

$R(\pi^* \Gamma, \Gamma)$ is finitely generated

if $K_X + \Delta$ is ~~Q-Cartier~~
fin. generated. $R(\pi_* \Delta)$

We fix the following situation:

Let $\pi : X \rightarrow U$ be a projective morphism of normal quasi-projective varieties where X has dimension n . Fix a general ample divisor A over U . Let V be an affine finite dimensional subspace of $W\text{Div}_U(X)$ which is defined over \mathbb{Q} . Suppose (X, Δ_0) is klt for some Δ_0 .

Assumption 

$$\mathcal{L}_A(V) = \left\{ \Delta = A + B \mid B \in V \quad B \geq 0 \right\}$$

$K_X + \Delta$ is lc

$$E_{A, \pi}(V) = \left\{ \Delta \in \mathcal{L}_A(V) \mid K_X + \Delta \text{ is pseudo-effective} \right\}$$

Assume then $C_n + D_n$ and
 assume \star . Then there are
 finitely many birational maps

$$\psi_j : X \dashrightarrow Z_j \text{ over } U$$

$1 \leq j \leq l$ s.t if

$$\psi : X \dashrightarrow Z \text{ is}$$

a weak log canonical model
 of $K_X + \Delta$ where $\Delta \in \mathcal{L}_A(V)$

then there is an index j

and there is an $\varrho : Z_j \rightarrow Z$

$$\text{s.t } \psi = \varrho \circ \psi_j.$$

Lemma 7.1 "weak finiteness of log terminal models".

Assume $C_1 + D_1$ and assumption * . Let $\mathcal{E} \subset \mathcal{L}_A(V)$ be a rational polytope s.t $\Delta \in \mathcal{E}$ then $K_X + \Delta$ is klt then there \exists finitely many birational maps

$\phi_i : X \dashrightarrow Y_i$ over U with the property that if $\Delta \in \mathcal{E} \cap \mathcal{E}_{A,II}(V)$ then there \exists an i s.t ϕ_i is a log terminal model for $K_{Y_i} + \Delta$.

Proof: w.l.o.g ℓ spans V_A .

Proof by induction on $\dim(\ell)$.

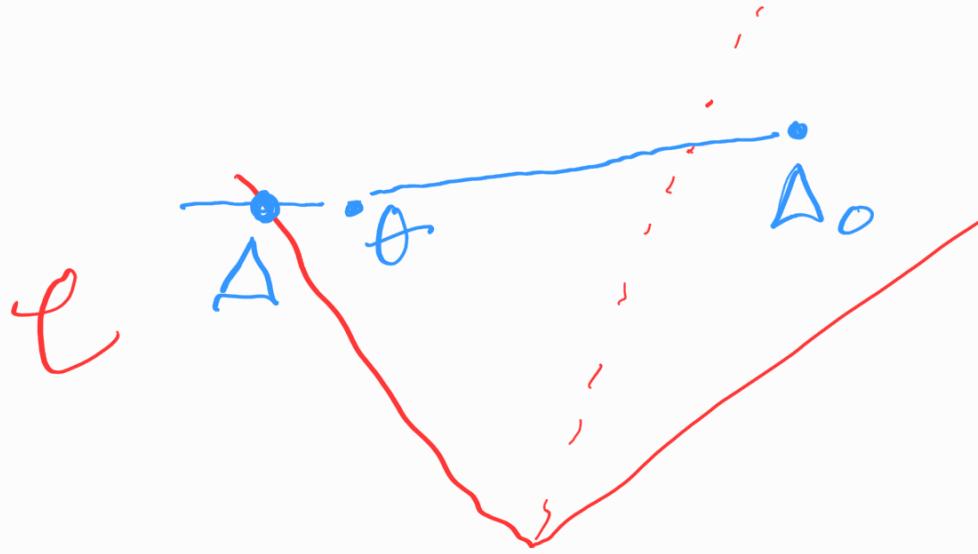
If $\dim(\ell) = 0$ if $\Delta_0 \in \ell \cap \mathcal{E}_{A,\Pi}^{(v)}$

then by Theorem $C_n + D_n$ there exists a log terminal model

$\phi_0: X \dashrightarrow Y_0$ of $K_X + \Delta_0$.

Assume that $\dim(\ell) \geq 0$.

"Idea of " proof under proof
assumption that there \exists
 $\Delta_0 \in \ell$ st
 $K_X + \Delta_0 \sim_{\mathbb{Q}, n} 0$



If $\theta \in \mathcal{E}$ then there exist a divisor Δ in the boundary of c and $\lambda \in (0, 1]$ s.t

$$\theta - \Delta_0 = \lambda(\Delta - \Delta_0).$$

$$\therefore \theta = \lambda\Delta + (1-\lambda)\Delta_0$$

$$K_X + \theta = \lambda(K_X + \Delta) + (1-\lambda)(K_X + \Delta_0)$$

$$\sim_{\mathbb{R}, U} \lambda(K_X + \Delta)$$

$$\theta \in \mathcal{E}_{\pi, A}(V) \Leftrightarrow \Delta \in \mathcal{E}_{\pi, A}(V)$$

l.t models for $K_X + \theta$ \hookrightarrow l.t models for $K_X + \Delta$

$\Delta \in \mathcal{E}'$ of $\dim \subset \dim(\mathcal{E})$
we are done by induction.

General Case:

Since we are working with R-divisors,
 $\epsilon \cap E_{A,\Pi}(V)$ is compact. It is therefore
sufficient to prove the statement
locally around $\Delta_0 \in \epsilon \cap E_{A,\Pi}(V)$.

Let $\phi: X \dashrightarrow Y$ be a l.t.-model
over U for $K_X + \Delta_0$ with
 $\Delta_0 \in \epsilon \cap E_{A,\Pi}(V)$.

Let $\Gamma_0 = \phi_* \Delta_0$, pick a small
Quotient map $\epsilon^0 \ni \Gamma_0$, s.t
for all $\Delta \in \epsilon^0$ $K_X + \phi_* \Delta$ is
l.t. Set $\epsilon' = \phi_*(\epsilon^0)$ we may
assume that $\epsilon' \subset S_{A'}(V')$.

w.l.o.g $K_X + \Delta_0$ is not over U .

$\Rightarrow K_X + \Delta_0$ is semiample
over U .

Let $f: X \rightarrow Z$ be the morphism defined by $R_X + \Delta_0$
 $R_X + \Delta_0 \underset{t, \mathbb{R}}{\sim} 0$. In particular
 by what was already proven
 there exists:
 finitely many birational maps
 ϕ_i
 $X \dashrightarrow Y_i$
 $\downarrow \begin{matrix} f \\ \downarrow \end{matrix} \downarrow$ such that
 for any $\Delta \in \mathcal{E}_A, f(\Delta)$ there
 $\exists \text{ an } i \text{ s.t. } \phi_i$ is a
 terminal model for $R_X + \Delta$ over
 Z .

It is sufficient to prove the following:

(or 3.11.3 section on Sheafes
Polytopes) (prelim. talk II)

In the setting above let ϕ_i be a log terminal model for $K_x + \Delta_0$ over \mathbb{Z} then there exists a nbh over \mathbb{Z} such that there exists a Δ s.t if $P \in \Delta_0$, $P \subset \mathcal{E}_{A,t}(V)$ s.t if $\Delta + P$ and ϕ_i is a $K_x + \Delta - l.t$ model over \mathbb{Z} then ϕ_i is a $K_x + \Delta - l.t$ model over \mathbb{U} .

Thus concludes the proof of the Lemma.

In preliminaries, lecture II we
 saw that the set of hyperplanes
 R^\perp inside $\mathcal{L}_A(V)$ is finite
 as R ranges over all
 extremal rays of $\overline{\text{NE}}(Y/n)$
 and that $N_{A,\text{II}}(V) = \{\Delta \in \mathcal{L}_A(V) : R_0 + \Delta\}$
 is a rational polytope

(Thm 3.11.1)

We will need also the second corollary
in the section on Shokrov Polytopes.

Cor 3.11.2 Fix assumption *.

Let $\phi: X \dashrightarrow Y$ be any birational
contraction over U .

Then

$W_{\phi, A, \Pi}(v)$ is a rational polytope
 $\{ \Delta \in E_{A, \Pi}(v) \mid \phi \text{ is a weak lc-model for } K_X + \Delta \}$
and there are finitely many
morphisms $f_i: Y \rightarrow Z_i$ over U ,
such that if $f: Y \rightarrow Z$ is any
contraction morphism with
 $K_Y + \Gamma = \underline{\phi(K_X + \Delta)} \sim_{f_* \mathbb{Q}} 0$ where
 $\Delta \in W_{\phi, A, \Pi}(v)$

then there is an index i
and an isom $\eta: Z_i \rightarrow Z$ s.t
 $f = \eta \circ f_i$.

We will now prove $C_n + D_n \Rightarrow E_n$

Lemma 7.2 Assume $C_n + D_n$ and assumption *. Let $\mathcal{C} \subset \mathcal{L}_A(v)$ be a rational polytope then there are finitely many birational maps

$$\Psi_j : X \dashrightarrow Z_j \text{ over } U$$

such that if

$\Psi : X \dashrightarrow Z$ is a weak lc-model of $K_X + \Delta$ over U for some $\Delta \in \mathcal{C}$ then there exists an index i and an isomorphism

$$\varsigma : Z_i \rightarrow Z \text{ s.t. } \Psi = \varsigma \circ \Psi_i$$

Remark: $\mathcal{C} = \mathcal{L}_{A,\Pi}(v)$ then this is theorem E_n .

Proof: Let $\psi: X \dashrightarrow \mathbb{Z}$ be a weak lc-model over u for $\Delta \in \mathcal{E}$

Step 1: Input: R-divisors so the question is local, $\Delta = A + B$. So we can perturb in order to make B contain any finite number of ample divisors in its support. [Reduce to the situation when there $\exists \Delta' \in \mathcal{E}_{A, \mathbb{H}}(v)$ s.t ψ is an ample model for $K_X + \Delta'$.]

Step 2: w.l.o.g $R_X + \Delta$ is klt
 $\forall \Delta \in \mathcal{E}$. Let $\Delta' \in \mathcal{E}_{\mathbb{A}, \mathbb{P}}(V)$
 be the divisor s.t
 $\psi: X \dashrightarrow Z$ is an ample
 model for $R_X + \Delta'$. By "finiteness"
 for l.t.-models there are
 finitely many brs. maps
 $\phi_i: X \dashrightarrow Y_i \quad 1 \leq i \leq r$
 s.t. \exists an index i s.t. ϕ_i
 is a log terminal model
 for Δ' . Let $R' = \phi_i^* \Delta'$ then
 $K_{Y_i} + R'$ is not $\Rightarrow K_{Y_i} + M'$
 big
 is semi-ample.

We can take an ample model

$y_i \xrightarrow{g'} z'_i$, by uniqueness
of ample model for $k_x + A'$

there is an isom $z' \cong z$

s.t

$$X \dashrightarrow Y_i \xrightarrow{\psi \circ g'} z$$

 ψ

let $\psi \circ g' = g$. By cor. 3.11.2

there are finitely many
morphisms

$$t_{i,m} : Y_i \rightarrow z_m \text{ such that}$$

s.t there exist an index
 m

and an isomorphism

$$\varsigma: \mathbb{Z}_m^l \rightarrow \mathbb{Z} \quad \text{s.t}$$

$$\mathcal{G} = \{\alpha_{i,j,m} \mid$$

$$\text{Let } \Psi_k: X \dashrightarrow Y_i \xrightarrow{\varphi_i} \mathbb{Z}_k^l$$

$$\text{then } \Psi = \varsigma \circ \Psi_k.$$

Proof of Step 1 We want to

Show that if $\Psi: X \dashrightarrow Z$ is a weak lc-model for $\Delta \in \mathcal{E}$ then there $\exists \Delta' \in \text{CNE}_{A,\mathbb{N}}(\mathcal{V})$ s.t. Ψ is an ample model for Δ' . W.l.o.g $\Delta \in \mathcal{E} \Rightarrow$ $K_{X+\Delta}$ is klt.

Let G be a divisor containing every divisor of V in its support. We take a log \mathbb{Q} -cscel. for (X, G)

$$f: Y \rightarrow X.$$

We use question is local so assume X is smooth, $\text{WDiv}_{\mathbb{R}}(X)$ is generated by (H_1, \dots, H_p) general ample divisors, $V \supset \text{Span}(H_1, \dots, H_p)$ and that every $\Delta \in \mathcal{E}$ contains $H_1 + \dots + H_p$ in its support.

Under these assumptions
if $X \xrightarrow{\psi} Z$ is a weak
lc-model for Δ then

$$\overline{W_{\phi, A, \Pi}}(v) = A_{\phi, A, \Pi}(v)$$

hence there \exists an $\Delta' \in \text{EN}_{A, f}$
s.t. ψ an ample model
for $K_X + \Delta'$. □

We have proven

$$C_n + P_n \Rightarrow E_n.$$

$$C_n + D_n + E_n$$



Theorem Fy

Let $\pi: X \rightarrow Z$ be a projective morphism to a normal affine variety. Let $(X, \Delta = A + B)$ be a klt pair of dimension n , A ample, $B \geq 0$. If $K_X + \Delta$ is pseudo-effective, then:

(1) (X, Δ) has a l.t. model & if $K_X + \Delta$ is \mathbb{Q} -Cartier then the log canonical ring is finitely generated.

Proof: Then $C_n + D_n \Rightarrow (1)$

(2) Let $V \subset W\text{Div}_R(X)$ be the vectorspace spanned by the components of Δ . Then there exists a constant $\delta > 0$ s.t if G is a prime divisor contained in the stable base locus of $K_X + \Delta$ and $E \in \mathcal{L}_A(V)$ is such that $\|E - \Delta\| \leq \delta$ then G is contained in the stable base locus of $K_X + E$.

Proof.: $\phi: X \dashrightarrow Y$ be l.f. and for $K_X + \Delta$, $\phi_* \Delta = \Gamma$ $K_Y + \Gamma$ is semi-ample. \Rightarrow every divisor G contained in the stable base locus of $K_X + \Delta$ was contracted by ϕ . For $\delta > 0$ small $\|E - \Delta\| \leq \delta$ then ϕ is still generic be $K_X + E$ negative. $\Rightarrow G$ is contained in the stable base locus of $K_X + E$.

③ Let WCV be the smallest affine subspace of $W\text{Div}_{\mathbb{R}}(X)$ containing Δ which is defined over \mathbb{Q} .

Then there is a constant $\eta > 0$ and a positive integer $r > 0$ s.t. if $\Sigma + W$ is a divisor with $\|\Sigma - \Delta\| < \eta$ and k any positive integer such that $\frac{k(R_x + \Delta)}{r}$ is Cartier, then every component of $\text{Fix}(k(R_x + \Sigma))$ is a component of the stable base locus of the stable base locus of $R_x + \Delta$.

$\phi: X \dashrightarrow S$ be a l.t model
 for $K_X + \Delta$, by theorem En
 for $\delta > 0$ small and
 $\|\Sigma - \Delta\| < \delta$ ϕ is a l.t
 -model for $K_X + \Sigma$.

$\phi_*(K_X + \Sigma)$ is semi-ample.
 There \nexists an r s.t it
 w $\phi_*(K_X + \Sigma)$ is coker
 then $r \in \phi_*(K_X + \Sigma)$ is
 base-point free.

if k is such that
 $\frac{k}{r}(K_X + \Sigma)$ is coker
 then $\phi_*(r \frac{k}{r}(K_X + \Sigma))$ is

b-se-point tree, and so
 $\text{Fix}(k(k_X + \sum))$ was contained
by 4. \Rightarrow Every component
of $\text{Fix}(k(k_X + \sum))$ is
(contained in) stable
base locus of
 $k_X + \Delta$.

Hence

$$C_n + D_n + E_n \Rightarrow F_n .$$